PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A
MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

On the Absolute Expansion of Mercury

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PHILOSOPHICAL TRANSACTIONS.

I. *On the Absolute Expansion of Mercury.*

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1. *Introduction.*

THE determination of the expansion of mercury by the absolute or hydrostatic method of balancing two vertical columns maintained at different temperatures does not appear to have been seriously attempted since the time of REGNAULT ('Mém. de l'Acad. Roy. des Sci. de l'Institut de France,' tome I., Paris, 1847). His results, though doubtless as perfect as the methods and apparatus available in his time would permit, left a much greater margin of uncertainty than is admissible at the present time in many cases to which they have been applied. The order of uncertainty may be illustrated by comparing the value of the fundamental coefficient of expansion (the mean coefficient between 0° and 100° C.) given by REGNAULT himself, with the values since deduced from his observations by WÜLLNER and by BROCH. They are as follows:—

REGNAULT	0·00018153.
WÜLLNER	0·00018253.
BROCH	0·00018216.

The discrepancy amounts to 1 in 180 even at this temperature, and would be equivalent to an uncertainty of about 4 per cent. in the expansion of a glass bulb determined with mercury by the weight thermometer method. The uncertainty of the mean coefficient is naturally greater at higher temperatures. If, in place of the mean coefficient, we take the actual coefficient at any temperature, the various reductions of REGNAULT'S work are still more discordant, and the rate of variation of the coefficient with temperature, which is nearly as important as the value of the mean coefficient itself in certain physical problems, becomes so uncertain that the discrepancies often exceed the value of the correction sought. It is only fair to REGNAULT to say that these discrepancies arise to some extent from the various assumptions made in reducing his results, and are not altogether inherent in the observations themselves.

The method of the weight thermometer permits an order of accuracy of about 1 in 20,000 in the determination of the weight of mercury expelled corresponding to the fundamental interval, but it necessarily leaves the absolute value of the fundamental coefficient uncertain, because it is obviously unfair to assume that the expansion of the containing bulb, however carefully annealed, is the same in all directions.

The recent determinations of the expansion of mercury by CHAPPUIS ('Travaux et Mémoires du Bureau International, 1907) by the weight thermometer method, employing a cylindrical bulb of *verre dur* of which the linear coefficient of expansion had been previously determined, gave results agreeing very closely between 0° C. and 100° C. with WÜLLNER'S reduction of REGNAULT'S observations. But as all the observations, with the exception of those at 100° C., were confined to the limits 0° C. and 44° C., the resulting equation for the expansion of mercury could not be applied with any confidence at temperatures above 100° C., especially as the values deduced from it differ by nearly 2·5 per cent. from REGNAULT'S at 300° C. The agreement with WÜLLNER'S reduction is possibly fortuitous, and the discrepancy from BROCH'S value of the fundamental coefficient might easily be explained by supposing that the expansion of the bulb employed by CHAPPUIS was about 2 per cent. less in the direction of its diameter than in the direction of its length, a supposition which is well within the limits of probability. It will be evident from the above summary that, in order to obtain trustworthy results for the cubical expansion of a bulb between 0° and 300° C., there was no alternative but to repeat REGNAULT'S method on a larger scale with modern appliances, the whole apparatus being designed, as far as possible, to give the same order of accuracy in the absolute expansion that is obtainable in the relative expansion by the weight thermometer method.

2. *General Description of the Apparatus and Method.*

The origin and progress of the present investigation has already been briefly sketched in two previous notes (Callendar, "Note on the Boiling-Point of Sulphur," 'Roy. Soc. Proc.,' A, vol. 81, p. 363, and Callendar and Moss, "The Boiling-Point of Sulphur corrected by reference to New Observations on the Absolute Expansion of Mercury," 'Roy. Soc. Proc.,' A, vol. 83, p. 106, 1909), and need not be repeated here. The general arrangement of the apparatus will readily be gathered from fig. 1, which shows a front and end elevation, and also a plan partly in section.

In place of the single pair of hot and cold columns, each 1·5 m. long, employed by Regnault, six pairs of hot and cold columns, each nearly 2 m. long, were connected in series, giving nearly eight times the expansion obtainable with Regnault's apparatus. The connections of the multiple manometer are indicated diagrammatically in fig. 2*a*. The hot and cold columns are marked H and C respectively, and occur alternately. If the mercury when in equilibrium stands at a in the gauge tube connected to the first cold column, and at z in the gauge tube connected to the last hot column, the difference of level to be measured, represented by a_1z, will be six times that due to a single pair of hot and cold columns. In the actual apparatus the cross tube *ef* was doubled back, so that *fg* lay behind *bc*, and *ih* behind *ed*, and so on, giving the arrangement represented in fig. 2*b*. All the hot columns were placed together in one limb EH (fig. 1) of a rectangle EFGH of iron tube, 5 cm. in bore, filled with circulating oil, and lagged with asbestos. All the cold columns were located in the corresponding limb BC of the similar rectangle ABCD. The outer limbs of the rectangles were utilized for the electrical heating coils FG, and the ice cooling bath WXYZ respectively. Centrifugal circulators continuously driven by an electric motor were provided for maintaining the oil in rapid circulation through the rectangles, so that the temperatures of the hot and cold columns were each nearly uniform. This arrangement of electric heating contributed greatly to the efficiency of the apparatus, as it produced the least possible disturbance of the surrounding conditions, and permitted the most easy and accurate regulation of the temperature.

The mean temperatures of the hot and cold columns were observed by means of a pair of platinum thermometers t_1 and t_2, contained in tubes similar in size to the tubes containing the mercury columns. The lengths of the loops of platinum wire forming the bulbs of the thermometers were made as nearly as possible equal to the lengths of the columns, and were fixed at the same level in the tubes, so as to give the true mean temperature, in case there were any appreciable variation throughout the length of the column.

The free ends of the series of hot and cold columns were connected, as indicated in fig. 1, by thick-walled rubber tubing to the glass tubes of the gauge, which is shown

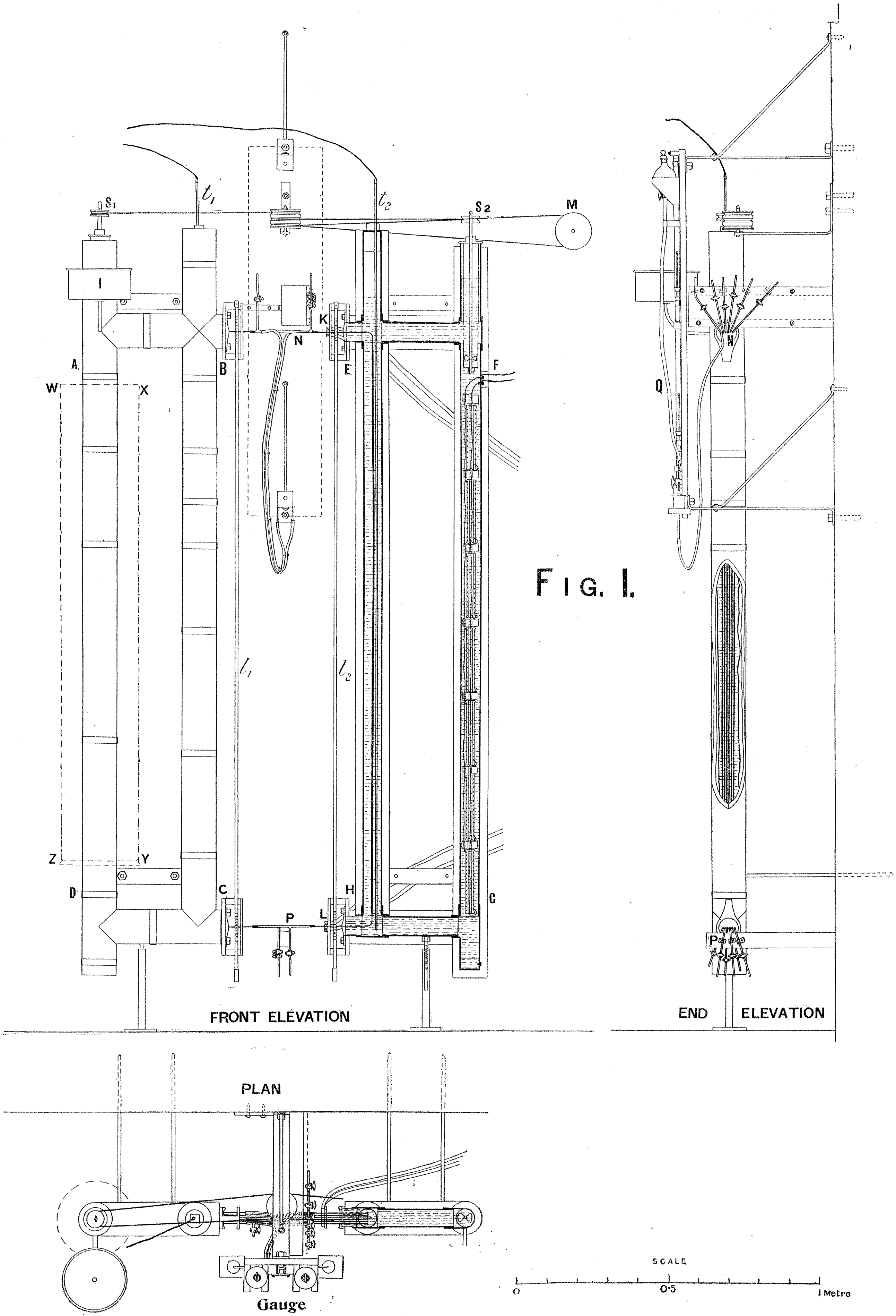

FIG. I.

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on a larger scale in fig. 4 (p. 6). The glass tubes of the gauge were 1·5 cm. in bore, and were fixed on either side of a standard invar metre of H section, to which the difference of level was directly referred by means of a pair of levelled telescopes, fitted with micrometer eye-pieces, and turning about a long vertical axis. The readings could be taken to 0·001 cm. The difference of level was about 20·5 cm. for a difference of temperature of 100° C., permitting an order of accuracy of 1 in 20,000.

As indicated roughly in fig. 2*b*, the length of the hot column l_2 was in general greater than the length of the cold column l_1, owing to the expansion of the iron tube rectangles containing the circulating oil. The expansion amounted to about

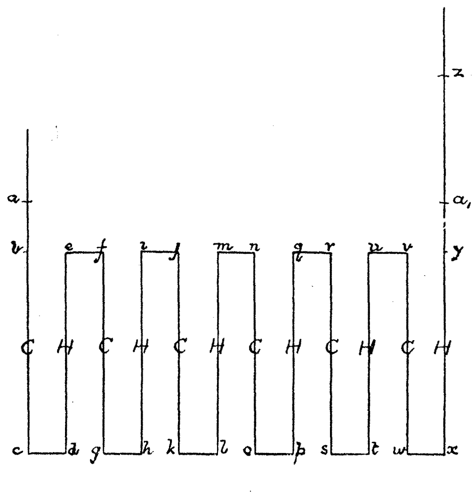

Fig. 2*a*.

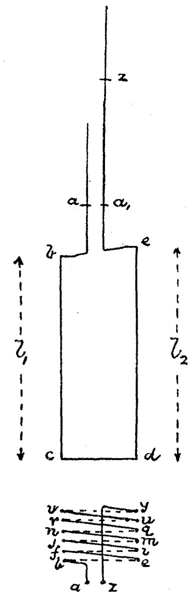

Fig. 2*b*.

8 mm. for 300° C., and necessitated a flexible connection between the hot and cold columns at the upper ends between the points *e* and *b*. The iron tube rectangles were firmly supported at the base, so that the lower cross tube *cd* was always very nearly horizontal. We are here concerned with the linear expansion only of the containing tubes, which will not affect the accuracy of the absolute values of the expansion of mercury, provided that adequate means are adopted for measuring the actual lengths of the hot and cold columns at each observation. The provision made by REGNAULT for this purpose was unsatisfactory, as he himself points out, especially in relation to his fourth series of observations, for which his apparatus was not originally designed. The essential point is that the tubes containing the mercury should be of small bore, and should be maintained accurately horizontal at the points where they emerge from the oil bath, and where the temperature changes from hot to cold. The method adopted for securing this result in the present investigation is shown in fig. 3. Steel tubes of 1 mm. bore were brazed with pure copper, using borax as a flux, into the upper and lower ends of the vertical steel tubes, 3·5 mm. bore, containing the mercury columns. The small bore tubes were bent round through a right angle, and were

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silver-soldered through holes accurately drilled in a brass plate A, which was clamped against the vertical face of the T joint in the tubes containing the circulating oil.

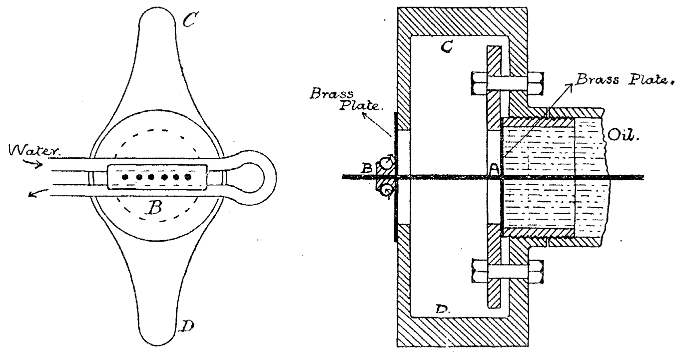

Fig. 3.

The mercury tubes up to the point A would thus be maintained at the temperature of the circulating oil. After traversing a distance of about 5 cm. in the air, the small bore tubes passed through a brass block B, being soldered into holes in the block drilled so as accurately to correspond with those in the plate A. The brass block B was carried by a rigid bracket CD, and was cooled by a water circulation as indicated in fig. 3. The plate A was adjusted so that the length of tube, AB, where the temperature changed from hot to cold, should be accurately horizontal. The same arrangement was adopted at each of the points, E, H, C, B, fig. 1, where the mercury tubes emerged from the circulating oil. The vertical heights of the hot and cold columns were measured by the steel tapes l_1, l_2, fig. 1, suspended from the upper brackets, and read by levelled telescopes. The effective heights of the columns were taken to be the vertical distances between the centres of the small bore steel tubes, which could be measured to about 0·1 mm., giving an order of accuracy of 1 in 20,000 in this fundamental measurement. The steel tapes were standardized by comparison with the standard invar scale, and were corrected for temperature at each observation.

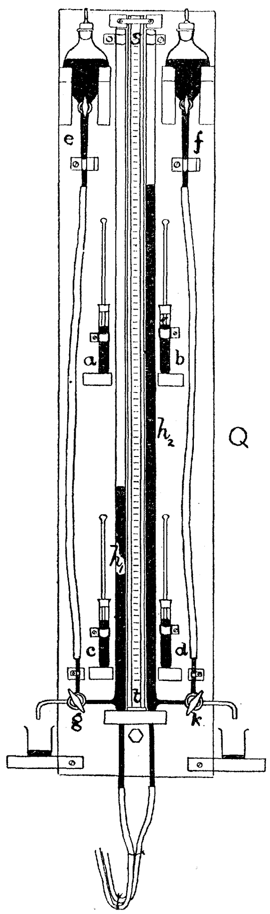

Fig. 4.

The gauge tubes, as shown in figs. 1 and 4, were mounted on a separate board in front of the apparatus so as to be protected from vibration and screened from the radiation of the hot columns. The temperature of the mercury in the gauge was estimated by means of four standardized mercury thermometers *a*, *b*, *c*, *d*, immersed in mercury contained in tubes of the same bore as the gauge tubes, and placed at a distance apart equal to twice the distance separating the gauge tubes. The difference of temperature between the gauge tubes was taken as half the mean difference of temperature between the thermometers in a horizontal direction. A correction for this difference of temperature was applied to the columns of mercury in the gauge tubes below aa_1, fig. 2*b*, *i.e.*, below the level of the top of the cold column. This correction never amounted to more than 0·002 cm., and is included in the recorded values of h given in the tables. The mean temperature of the column representing the difference of level h was estimated from the vertical and horizontal temperature gradients indicated by the thermometers, and is denoted by t in the tables and equations. The accuracy required in the observation of the gauge temperature was about fifty times less than in the temperatures of the hot and cold columns.

The platinum thermometers were annealed in place in the apparatus by heating the whole to 350° C. shortly after its erection before filling with mercury. This annealing reduced the resistance of each by 1 part in 3000. The apparatus was not heated afterwards beyond 300° C., and the thermometers showed no signs of further change. Owing to their great length the thermometers could not be tested satisfactorily, except at 0° and 100° C., and the value of the difference coefficient 0·000150 was assumed to be the same as that found for other thermometers constructed of the same wire. The fundamental intervals of the thermometers were, for t_1, 4·6456, and for t_2, 4·6412 ohms. Readings were taken to 0·1 mm. on the bridge wire, corresponding to 0°·002 C., giving an order of accuracy of 1 in 50,000 on the fundamental interval. The values of t_1 and t_2 recorded in the tables were deduced from the observed temperatures on the platinum scale pt_1, pt_2, by the formula,

$$t - pt = 0{\cdot}000150\,t\,(t-100),$$

which may possibly be in error by 0°·03 C. (or 1 in 10,000) at 300° C.

3. *Theory and Notation.*

The following notation is adopted :—

$H_1 = 6l_1$ is the effective height of the cold column at a temperature t_1.
$H_2 = 6l_2$ is the effective height of the hot column at a temperature t_2.
$dH = H_2 - H_1$ is the effective height of the cross tubes at the air temperature t.
h is the observed difference of level in the gauge tubes at the temperature t.
$h' = h - 0{\cdot}00018\,(h-dH)\,(t-t_1)$ is h corrected for dH and reduced to t_1.

The error of this reduction will not exceed 1 in 20,000 of h provided that the temperature of the gauge t is known within 0°·3 C., and that the temperature of the cross tubes does not differ by more than 4° C. from the gauge. The approximate value 0·000180 of the coefficient of expansion suffices within the same limits of accuracy provided that the difference of temperature $t - t_1$ does not exceed 50° C. The temperature of the gauge may be taken as known within 0°·1 C. The cross tubes seldom differed in temperature from the gauge by so much as 1° C., and the difference of temperature $t-t_1$ in the majority of the observations was less than 1° C. The required correction to the value of h was therefore extremely small, and introduced very little uncertainty in the reduction.

Considering the equilibrium of the columns, we have the hot column H_2 at a temperature t_2, together with the difference of level h in the gauge at a temperature t, balanced by the cold column H_1 at a temperature t_1, together with the cross tubes dH at a temperature t. The mean coefficient of expansion ${}_1\alpha_2$ between t_1 and t_2 in terms of the volume at t_1, which is the coefficient most directly given by the observations, is easily obtained by reducing the columns to a common temperature t_1. We thus obtain the equation

$$H_2/\{1+{}_1\alpha_2(t_2-t_1)\}+(h-dH)/\{1+0{\cdot}00018(t-t_1)\} = H_1 = H_2-dH, \quad . \quad . \quad (1)$$

which, with a few simple approximations in the small terms involving h, reduces to

$${}_1\alpha_2(t_2-t_1) = h'/(H_2-h'), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where h' denotes the corrected and reduced value of h given above.

The expansion between t_1 and t_2 may further be expressed as a fraction of the volume at 0° C. by multiplying by the factor $(1+0{\cdot}00018t_1)$. The uncertainty of this reduction will not exceed 1 in 40,000 unless t_1 exceeds 25° C. Thus,

$$({}_1\alpha_2)_0(t_2-t_1) = (1+0{\cdot}00018t_1)\,h'/(H_2-h'), \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where $({}_1\alpha_2)_0$ denotes the mean coefficient t_1 to t_2 in terms of the volume at 0° C.

The expansion between 0° C. and t_2 in terms of which the results are generally tabulated, may readily be deduced by adding the expansion between 0° C. and t_1 to that between t_1 and t_2 in terms of the volume at 0° C. Thus,

$${}_0\alpha_2t_2 = {}_0\alpha_1t_1+({}_1\alpha_2)_0(t_2-t_1), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

but this involves a correction of quite a different order of magnitude, and requires the value of ${}_0\alpha_1$ to be accurately known, unless t_1 is very small.

The two last reductions may be included together in the formula

$${}_0\alpha_2t_2 = (h'+{}_0\alpha_1t_1H_2)/(H_2-h'), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

but since the correction term ${}_0\alpha_1t_1H_2$ may be nearly as large as h', it is desirable to keep this correction separate from the others, and to make a special series of observations to determine it.

This important point is somewhat obscured in REGNAULT'S formula, and has led to considerable uncertainty in the reduction of his results. The majority of his observations were taken with the cold column at a temperature in the neighbourhood of 20° C. It makes a difference of more than 1 in 500 in the fundamental coefficient according as we assume REGNAULT's value 0·0001795, or WÜLLNER's value 0·0001814, for the mean coefficient between 0° C. and 20° C. in reducing the observations. The uncertainty is greater at lower temperatures. REGNAULT states that he solved his formula by a method of successive approximation, but the approximation would evidently be unsatisfactory at low temperatures, and his calculations cannot be reproduced so as to make his results fit with his observations. REGNAULT himself was conscious of this difficulty, and endeavoured to avoid it by cooling the cold column with melting ice, but he appears to have abandoned this method on account of difficulties of manipulation. The apparatus employed in the present investigation was better suited for the purpose than REGNAULT's, and a special series of observations was successfully taken with the cold column in ice, and at −10° C., for the accurate determination of the coefficient at low temperatures. But the majority of the observations were taken with the cold column at the atmospheric temperature, because this procedure, besides greatly facilitating the manipulation, made all the other corrections as small as possible, and in particular rendered the correction depending on dH practically negligible, so that it was in most cases unnecessary to measure the length of the cold column at each observation.

It is easily seen that a formula precisely analogous to (5) applies to the reduction of the observations to any convenient standard temperature t_0, other than 0° C., namely,

$$_0\alpha_2\,(t_2-t_0) = [h'+{}_0\alpha_1\,(t_1-t_0)\,\mathrm{H}_2]/(\mathrm{H}_2-h'), \qquad (6)$$

where $_0\alpha_2$, $_0\alpha_1$, denote the mean coefficients between t_0 and t_2, t_1 respectively expressed in terms of the volume at t_0. Formulæ (2) and (5) may be regarded as special cases of this more general formula in which t_0 is replaced by t_1 and by 0° C. respectively.

The majority of the observations in the first series with the cold column at the atmospheric temperature in the neighbourhood of 20° C., were reduced to a standard temperature of 20° C. in the first instance, because the value of the coefficient at 20° C. in terms of the volume at 20° C. could be inferred with considerable accuracy from the observations themselves, and the difference (t_1-20) was comparatively small. The correction term $(t_1-20)\,{}_0\alpha_1\mathrm{H}_2$ was of the order of 3 per cent. at most, and was itself known with certainty to 1 in 2,000. If the observations had been reduced directly to 0° C. by REGNAULT's formula, this correction would in some cases have exceeded 30 per cent., and would have been most uncertain, since the mean coefficient from 0° C. to 20° C. could be obtained only by extrapolation. The corrections involved in deducing h' from h, were of the order of 2 or 3 parts in 10,000 only, and could not give rise to any similar uncertainty.

With the apparatus above described, the expansion of mercury is obtained under a mean pressure of 2·5 atmospheres, but the result will not differ from the expansion under a pressure of 1 atmosphere except in so far as the compressibility of mercury varies with temperature. The compressibility of mercury, however, is so small that it would require a variation of 50 per cent. to affect the results appreciably even at 300° C. It is most improbable that the variation of compressibility with temperature is as great as this. It would have been necessary to apply a pressure of 2 or 3 atmospheres to the gauge to test this point satisfactorily, and it was not considered advisable to do this, since any accidental failure of any of the joints or taps under pressure at high temperatures might have involved dismounting and filling the whole apparatus afresh, and would have seriously interfered with the continuity of the observations.

4. *Method of Filling the Apparatus.*

The adoption of the multiple manometer method, which was rendered necessary in order to avoid excessive length and pressure, entailed some difficulty in filling the apparatus. After adjusting the tubes in position, as shown in fig. 1, the small bore steel tubes at the top and bottom of the hot and cold columns were connected by horizontal glass tubes, as shown in fig. 5*a*, with taps attached at right angles. The glass tubes nearly fitted the steel tubes, and the joints were made tight by running in a mixture of beeswax and resin. The taps were suitably supported, and projected fanwise, upwards at the top, and downwards at the bottom, as shown in the side view in fig. 1. After the taps had been connected and the whole tested for leaks, the apparatus was heated to about 350° C., and evacuated, and dried by passing filtered air through it. When cool, the apparatus was evacuated as completely as possible, and mercury was admitted by connecting a reservoir to each of the lower taps in turn, until the level of the mercury rose nearly to the upper cross tubes. The apparatus was again evacuated by connecting the pump to each of the upper

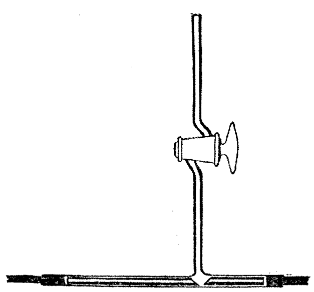

Fig. 5*a*.

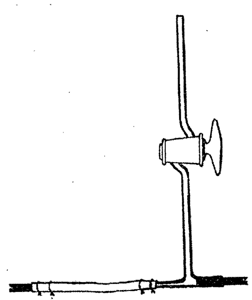

Fig. 5*b*.

taps in turn, in order to remove any air displaced by the mercury. The filling was then completed and the gauge tubes connected. The absence of air was shown by the fact that, if the level on one side were disturbed by running mercury into or out of the gauge, by means of the three-way taps g, k, and mercury reservoirs e, f (fig. 4) provided for the purpose, an equal change of level was almost immediately apparent on the other side of the gauge. If the level was raised about 1 cm. by inserting a glass plunger, without withdrawing or adding mercury, the levels returned to their previous values within 0·01 mm. in about a minute after the removal of the plunger, in spite of the great length of fine tube through which the mercury had to flow. If, on the other hand, the continuity of the mercury column was broken by a single air bubble in one of the fine tubes, the level could be altered by a centimetre or more on one side without any change taking place on the other. It was feared at first that the presence of air bubbles might be a serious source of error, but the effect produced was so immediately obvious that no uncertainty arose in this way.

In spite of the care taken in evacuating the apparatus, some bubbles invariably appeared when the apparatus was first heated to high temperatures such as 200° C. to 300° C. after each fresh filling. These bubbles were removed as they appeared by altering the level of the mercury in the gauge, so as to reduce the pressure and drive the bubbles round into the open space where the tap was connected, whence they could be removed by applying the air pump.

At the highest temperatures, from 200° C. to 300° C., it was found necessary on account of the expansion of the containing tubes, which amounted to nearly 8 mm. at 300° C., to insert a flexible rubber connection, as indicated in fig. 5*b*, between the glass taps and the small-bore tubes on the cold side. The T-joint was placed close to the end of the small-bore steel tube on the hot side to facilitate the trapping of bubbles, which were most troublesome at the higher temperatures. Fortunately this trouble tended to disappear as the occluded or dissolved gas was removed by repeated heating of the mercury.

5. *Method of Taking Observations.*

After starting the water and oil circulations, the heating coils were connected to the electric-light mains. A current of 13 amperes at 100 volts sufficed to raise the temperature about 100° C. in an hour and a half. When the temperature approached the required point, the current was gradually reduced to the value which experience had shown to be sufficient to maintain the desired temperature. The current was then switched over to a large battery of accumulators, which made it possible to keep the temperature very nearly constant with slight occasional adjustments. But as the cold columns rose very slowly in temperature, at the rate of about a tenth of a degree in half an hour, the current was generally set to give a slightly greater rate of rise for the hot column during half an hour or so, followed by a slightly slower rate for another half hour, so that during the first period the difference of temperature and the difference of level might be slowly increasing, and during the second period slowly diminishing, at nearly the same rate. By taking observations in this way the effects of lag, if any, would be eliminated from the results. The actual readings of temperature and of difference of level were plotted on a large scale (10 cm. to 4 cm. on the bridge wire and 10 cm. to 2 mm. of the difference of level), on a time base, so that the conditions of the experiment could be accurately followed, and any defect in the working of the apparatus, such as the appearance of an air

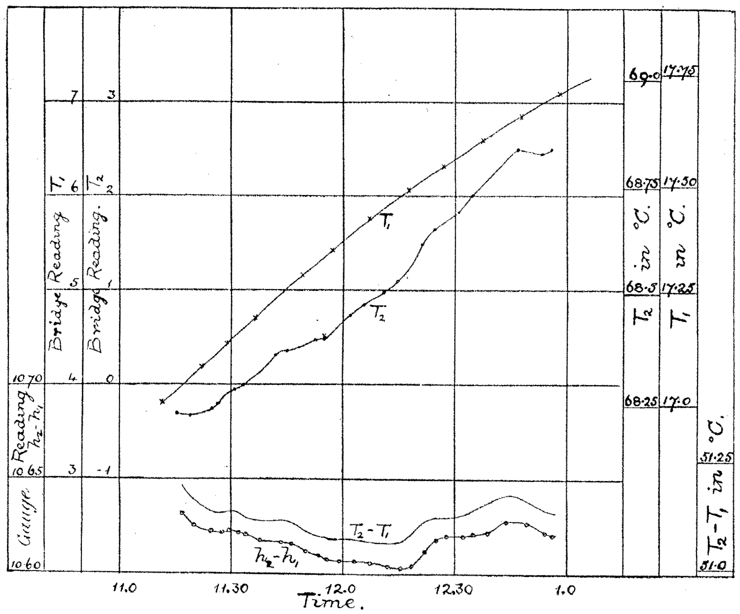

Fig. 6.

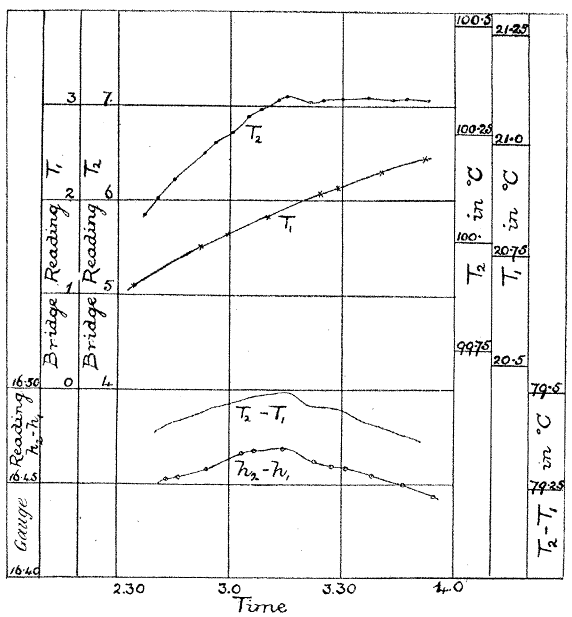

Fig. 7.

bubble at high temperatures, immediately detected. The curves shown in fig. 6 and fig. 7 are typical examples, and will suffice to show how closely the difference of level $h_2 - h_1$ followed every change in the difference of temperature $T_2 - T_1$. The same

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curves also served as a convenient means of graphic interpolation for deducing the simultaneous values of the observed quantities.

Since all the readings could not be made simultaneously, attention was directed to obtaining them in quick and regular succession. The bridge was provided with mercury cup connections in place of the usual plugs, giving great improvement in quickness as well as in accuracy. The thermometers could be interchanged instantaneously by means of a mercury cup switch, without introducing any variable contact errors such as would have been unavoidable with screw connections. The telescopes for reading the mercury levels contained eyepiece micrometers divided to tenths of a millimetre, fitted with vertical screw adjustments, and were focussed and adjusted so that the millimetre divisions coincided with those of the standard invar scale situated between the mercury columns. By suitably shading and illuminating the mercury columns, readings could easily be taken by inspection to 0·01 mm. A separate handle was provided for turning the vertical column carrying the telescopes through an angle of 2 degrees either way to verify the adjustment of the eyepiece micrometers on the invar scale. This method appeared greatly preferable in practice to the use of a filar micrometer, since it was never necessary to touch the telescopes when once they had been adjusted for a run. With a little practice, a single observer could take all the readings, and perform all the necessary adjustments, without any excessive haste or exertion.

In reducing the observations, points were selected on the curves both with a rising and falling temperature difference, where the temperature conditions appeared to be most favourable. Points taken on the same day at nearly the same temperature always agreed so closely on reduction, that it was considered preferable to take short runs of about one hour each at two or three different temperatures on the same day rather than runs of long duration at one temperature. Runs taken at the same temperature on different days, separated often by many months, when all the conditions of observation were completely changed, afforded a much better test of the accuracy of the method, and were more likely to serve for the elimination of constant or accidental errors than runs taken under constant conditions. Observations given in the tables under the same date at the same temperature were taken with a rising and falling temperature respectively, or otherwise differed materially in the conditions under which they were taken.

6. *Method of Reducing the Observations.*

The following example, showing the reduction of a single observation, will serve to illustrate the order of magnitude of the corrections involved. The corrections were worked to one figure beyond the limit of accuracy of reading, except that, in the case of the platinum thermometers reading to 0°·002 C., it was considered useless to express the temperatures beyond 0°·001 C., as this represented an order of accuracy five times as great as could be obtained in reading the difference of level on the gauge :—

November 2, 1908. 3 p.m. Current 7·52 amperes.	Temperature difference increasing.	
Box temperature observed, 21·0° C.	Cold side.	Hot side.
Readings of platinum thermometers	1291·71	1656·82
Calibration and temperature corrections	+0·153	+0·228
Corrected bridge readings, R	1291·863	1657·048
Resistances at 0° C., R_0	1194·052	1191·657
Differences, $R-R_0$	97·811	465·391
Fundamental intervals, $R_{100}-R_0$	464·56	464·12
Temperatures on platinum scale, pt	21·055	100·274
Corrections to gas scale $(t-pt)$	−0·247	+0·004
Temperatures t_1 and t_2 on gas scale	20·808	100·278

Mercury gauge readings :—	Cold side.	Hot side.
Levels of mercury in gauge (cm. corrected)	51·1522	67·6196
Upper thermometers (at 52 cm. level)	21°·6 C.	21°·9 C.
Lower thermometers (at 8 cm. level)	20°·95 C.	21°·0 C.
Mean temperature of columns below 51 cm.	21°·27 C.	21°·35 C.
Correction for temperature difference, 0°·08 C.	−0·0009 cm.	
Difference of level, h = 67·6196−51·1522 =	16·4674 cm.	
Mean temperature of h, 22°·0 C. Correction of h to t_1	−0·0035	

Observed lengths of columns by steel tapes corrected	192·755	192·815

Effective height, $H_2 = 6l_2 = 1156·890$ cm. $dH = H_2-H_1 = 0·360$ cm.

Temperature of dH observed, 21°·1 C. Reduction to t_1 negligible (0·00002 cm.).

Corrected value of h, $h' = h-0·00018\,(h-dH)\,(t-t_1) = 16·4630$.

Expansion t_1 to t_2, ${}_1\alpha_2\,(t_2-t_1) = h'/(H_2-h') = 0·0144358$.

Mean coefficient t_1 to t_2 in terms of volume at t_1, ${}_1\alpha_2 = 0·000181645$.

To facilitate comparison between the different observations in which the cold column was at atmospheric temperature, the results were further reduced to a standard temperature at 20° C. for the cold column, assuming the value of the coefficient at 20° C. in terms of the volume at 20° C. to be 0·0001805. The value of this coefficient could be deduced, with sufficient accuracy for the purpose, from the first series of observations, extending from 20° C. to 187° C. This reduction may be effected by means of formula (6), p. 9, and involves the addition to h', in the numerator of the fraction representing the expansion of a quantity 0·0001805 × 1156·89 × 0·808 = 0·1688 cm., which amounts in the present instance to about 1 per cent. of h, but is known with considerable accuracy. Including this correction we have finally :—

Expansion from 20° C. to 100°·278 C. in terms of volume at 20° C.

$$ {}_{20}\alpha_2 (t_2 - 20) = \frac{16{\cdot}6318}{1140{\cdot}427} = 0{\cdot}0145838. $$

Mean coefficient from 20° C. to 100·278° C. in terms of volume at 20° C.

$$ {}_{20}\alpha_2 = \frac{0{\cdot}0145834}{80{\cdot}278} = 0{\cdot}000181665. $$

The further reduction to 0° C., involving a correction of 20 per cent., could not be effected satisfactorily until the conclusion of the third series of observations, and was not required for comparing the results of the first two series, which are therefore reduced to 20° C. in the tables given below.

The corresponding observation taken on the same day, with the difference of temperature *decreasing*, was as follows:—

November 2, 1908, 3.30 p.m. Current, 7·50 amperes. Resistance box, 21° C.

Bridge readings corrected for calibration and temperature	1292·436	1657·310
Temperatures on gas scale deduced . . .	t_1 = 20°·930 C.	t_2 = 100°·335 C.
Lengths on hot and cold columns	l_1 = 192·755	l_2 = 192·815
Levels of mercury in gauge (corrected for scale)	h_1 = 51·1552	h_2 = 67·6102

Gauge thermometers: upper 21°·35 C., 21°·70 C.; lower 20°·9 C., 21° C.
Temperatures of cross tubes: upper 20°·7, lower 20° C.
Corrected difference of level, h' = 16·4515 cm.
Temperature of h, t = 21°·72 C.
Reduction to 20° C. = $0{\cdot}0001805 \times 0{\cdot}930 \times H_2 = +0{\cdot}1942$ cm.

$$ {}_{20}\alpha_2 (t_2 - 20) = \frac{16{\cdot}6457}{1140{\cdot}438} = 0{\cdot}0145958. $$

$$ {}_{20}\alpha_2 = \frac{0{\cdot}0145958}{80{\cdot}335} = 0{\cdot}000181685. $$

The difference from the first observation would be explained by a lag of 0·001 cm. either way in the gauge reading, but is within the limits of accuracy of observation.

7. *Summary of Observations.*

The following tables contain a summary of all the observations taken after the apparatus had been got into proper working order. Observations taken with the same upper limit of temperature t_2 are grouped together to facilitate comparison, and the observations in each group are arranged in order of date. The first column gives the date. The second column gives the observed value of the effective height of the hot column H_2 corrected for scale error and temperature. The correction for the difference of length dH of the hot and cold columns was always negligible in the first two series, and the value of dH is not given in the tables. In the third series this correction became appreciable, and a separate column is added giving the values of dH. The third and fourth columns give the temperatures t_1 and t_2 of the cold and hot columns, reduced to the gas scale. The fifth column gives the value of the difference of level h in the gauge, corrected for errors of the standard invar scale, and for difference of temperature between the gauge columns. These corrections seldom exceeded 0·001 cm., the limit of accuracy of reading and the data for applying them could not have conveniently been included in the tables. The value of h is not corrected for the mean temperature of the column h itself, which is given under the heading t in the next column. The seventh column contains the value of the expansion $_{20}\alpha_2(t_2-20)$ from 20° C. to t_2 in terms of the volume at 20° C., calculated by formula (6), to the same order of accuracy as the values of h, namely, to one figure beyond the limit of accuracy of reading. The values of the expansion are not directly comparable, because they include the small variations of t_2. The last column is accordingly added, giving the corresponding value of the mean coefficient $_{20}\alpha_2$ from 20° to t_2 in terms of the volume at 20° C. A variation of a tenth of a degree in t_2 should produce a variation of about 2 in the last figure of this coefficient, so that the small variations of t_2 would seldom affect the last figure but one of the coefficient. The differences shown in this column exhibit the accumulated effect of all the possible errors of observation, including the effect of lag, to which many of the larger differences appear to be due. Since most of the observations were purposely taken in pairs, as explained above, in such a way as to exhibit this effect, with a view to detecting and eliminating it, it is probable that the accuracy of the final means is not seriously affected by this source of error.

Series I.—Observations, 20° C. *to* 187° C.

Date.	H_2.	t_1.	t_2.	h.	t.	$_{20}\alpha_2(t_2-20)$.	$_{20}\alpha_2$.
			(1) Observations at 68°·5 C.				
1908							
Oct. 24 . .	1156·53	18·335	68·766	10·4723	18·89	·0088335	·000181144
,, 24 . .	1156·53	18·690	69·070	10·4598	18·92	·0088879	·000181127
Nov. 13 . .	1156·47	17·993	68·693	10·5307	18·87	·0088225	·000181186
,, 13 . .	1156·47	18·418	68·755	10·4568	18·94	·0088356	·000181224
Nov. 14 . .	1156·38	18·200	68·506	10·4503	18·74	·0087907	·000181229
,, 14 . .	1156·38	18·359	68·766	10·4685	18·64	·0088361	·000181194
,, 14 . .	1156·38	18·119	68·469	10·4608	18·77	·0087851	·000181249

Series I.—Observations, 20° C. *to* 187° C. (continued).

Date.	H_2.	t_1.	t_2.	h.	t.	${}_{20}\alpha_2(t_2-20)$.	${}_{20}\alpha_2$.
(1) Observations at 68°·5 C. (continued).							
1908							
Nov. 16 . .	1156·41	17·249	68·371	10·6175	17·90	·0087643	·000181189
„ 16 . .	1156·41	17·584	68·736	10·6223	17·94	·0088300	·000181180
Nov. 16 . .	1156·41	17·921	68·569	10·5176	18·21	·0087993	·000181171
„ 16 . .	1156·41	18·019	68·695	10·5228	18·33	·0088219	·000181166
Nov. 18 . .	1156·41	17·072	68·438	10·6714	17·72	·0087795	·000181252
„ 18 . .	1156·41	17·268	68·610	10·6619	17·91	·0088068	·000181172
Means =	—	—	68·649	—	—	·0088148	·000181192
(2) Observations at 100° C.							
1908							
*Oct. 31 . .	1156·77	20·065	100·278	16·6248	21·42	·0145895	·000181742
„ 31 . .	1156·77	20·416	100·350	16·5689	21·34	·0146053	·000181771
*Oct. 31 . .	1156·77	21·153	100·278	16·3992	21·60	·0145905	·000181750
„ 31 . .	1156·77	20·883	100·350	16·4724	21·60	·0146055	·000181777
†Nov. 2 . .	1156·89	20·081	100·278	16·6134	20·80	·0145825	·000181650
„ 2 . .	1156·89	19·885	100·278	16·6596	20·60	·0145878	·000181716
†Nov. 2 . .	1156·89	20·807	100·278	16·4665	22·00	·0145837	·000181664
„ 2 . .	1156·89	20·930	100·335	16·4539	21·72	·0145958	·000181685
Nov. 4 . .	1156·83	19·192	100·353	16·8160	20·50	·0145992	·000181689
„ 4 . .	1156·83	19·557	100·344	16·7395	20·77	·0145984	·000181697
Nov. 6 . .	1156·80	19·002	100·463	16·8820	20·57	·0146227	·000181732
„ 6 . .	1156·80	19·589	100·342	16·7305	20·49	·0145973	·000181690
Nov. 7 . .	1156·89	18·240	100·159	16·9745	19·37	·0145655	·000181707
„ 7 . .	1156·89	18·575	100·236	16·9181	19·32	·0145778	·000181686
Nov. 9 . .	1156·92	16·947	100·441	17·2950	18·07	·0146135	·000181667
„ 9 . .	1156·92	17·231	100·260	17·2001	18·01	·0145820	·000181684
Means =	—	—	100·314	—	—	·0145936	·000181707

* The first pair of observations on this date was taken under a pressure of 45 cm. of mercury more than the second.

† The first pair of observations on this date was taken under a pressure of 40 cm. of mercury less than the second.

Series I.—Observations 20° C. *to* 187° C. (continued).

Date.	H_2.	t_1.	t_2.	h.	t.	${}_{20}\alpha_2\,(t_2-20)$.	${}_{20}\alpha_2$.
			(3) Observations at 116°·5 C.				
1908							
Oct. 20 . .	1157·07	18·774	116·476	20·2186	20·12	·0175551	·000181963
„ 20 . .	1157·07	19·290	116·474	20·1193	20·20	·0175624	·000182043
„ 20 . .	1157·07	19·822	117·082	20·1308	21·12	·0176692	·000182003
Nov. 20 . .	1157·13	17·161	116·954	20·6536	18·17	·0176482	·000182026
„ 20 . .	1157·13	17·367	117·272	20·6704	18·19	·0177019	·000181983
„ 20 . .	1157·13	16·781	116·438	20·6271	17·90	·0175543	·000182027
Nov. 21 . .	1157·10	16·993	117·009	20·6930	18·04	·0176532	·000181975
„ 21 . .	1157·10	17·391	116·892	20·5900	18·30	·0176344	·000182002
„ 21 . .	1157·10	17·560	117·122	20·5976	18·32	·0176728	·000181965
Means =	—	—	116·858	—	—	·0176279	·000181997
			(4) Observations at 150° C.				
1908							
Nov. 23 . .	1157·58	17·729	150·707	27·4721	19·02	·0238836	·000182726
„ 23 . .	1157·58	18·554	150·563	27·2600	19·36	·0238462	·000182641
Nov. 27 . .	1157·58	17·939	150·148	27·3082	19·20	·0237741	·000182670
„ 27 . .	1157·58	18·080	150·351	27·3232	19·31	·0238140	·000182691
„ 27 . .	1157·58	18·326	150·540	27·3125	19·47	·0238501	·000182703
Nov. 28 . .	1157·58	18·047	150·021	27·2692	19·53	·0237578	·000182723
„ 28 . .	1157·58	18·271	150·308	27·2788	19·62	·0238085	·000182709
Means =	—	—	150·375	—	—	·0238192	·000182697
			(5) Observations at 187° C.				
1909							
Jan. 21 . .	1158·18	16·624	187·325	35·1804	17·63	·0306926	·000183431
„ 21 . .	1158·18	16·751	187·487	35·1884	17·77	·0307236	·000183439
Feb. 27 . .	1158·30	17·341	187·660	35·1125	18·66	·0307589	·000183460
„ 27 . .	1158·30	17·393	187·669	35·1039	18·38	·0307625	·000183472
Means =	—	—	187·535	—	—	·0307344	·000183451

Shortly after the conclusion of the last observations at 187° C., in attempting to take an observation at a temperature above 200° C., the mercury was observed to be falling in the gauge at the rate of about a tenth of a millimetre in 10 minutes. When the apparatus was cold, the level continued to fall at the rate of about 1 cm.

per day. A leak in one of the copper-brazed joints was suspected, but on dismounting the apparatus it was found that one of the solid drawn steel tubes had apparently split in the process of manufacture, and had been brazed up with ordinary spelter by the makers so skilfully that the flaw had escaped detection when the apparatus was put together. In process of time the hot mercury had naturally found its way through the brass. A completely new set of steel tubes was accordingly fitted, which occasioned a good deal of delay. Owing to the pressure of other duties, the apparatus could not be got ready for work again till the end of June.

Series II.—Observations, 187° C. *to* 300° C.

Date.	H_2.	t_1.	t_2.	h.	t.	${}_{20}\alpha_2\,(t_2-20)$.	${}_{20}\alpha_2$.
			(1) Observations at 187° C.				
1909							
June 28 .	1160·97	19·309	187·442	34·7558	20·28	·0307267	·000183507
,, 28 .	1160·97	19·636	187·941	34·7894	20·77	·0308173	·000183501
July 9 .	1160·97	20·861	187·001	34·3608	21·60	·0306553	·000183563
,, 9 .	1160·97	21·007	187·153	34·3600	21·80	·0306814	·000183553
Means =	—	—	187·384	—	—	·0307202	·000183531
			(2) Observations at 221° C.				
1909							
June 28 .	1161·51	21·435	221·365	41·2574	22·91	·0370874	·000184180
,, 28 .	1161·51	21·203	221·122	41·2592	22·53	·0370463	·000184198
July 9 .	1161·51	22·567	221·049	40·9783	22·89	·0370484	·000184276
,, 9 .	1161·51	22·890	221·026	40·9161	23·49	·0370496	·000184303
July 12 .	1161·54	22·039	221·162	41·1244	23·61	·0370753	·000184306
Means =	—	—	221·145	—	—	·0370617	·000184253
			(3) Observations at 260° C.				
1909							
July 14 .	1162·17	23·951	260·041	48·6868	25·80	·0444540	·000185193
,, 14 .	1162·17	24·235	260·041	48·6286	26·30	·0444512	·000185182
July 20 .	1162·17	27·771	259·734	47·8494	27·60	·0444048	·000185225
,, 20 .	1162·17	27·896	259·918	47·8584	27·70	·0444369	·000185217
July 28 .	1162·17	22·659	259·785	48·9016	24·93	·0444085	·000185201
Means =	—	—	259·904	—	—	·0444308	·000185202

Series II.—Observations, 187° C. *to* 300° C. (continued).

Date.	H_2.	t_1.	t_2.	h.	t.	${}_{20}\alpha_2(t_2-20)$.	${}_{20}\alpha_2$.
(4) Observations at 300° C.							
1909 July 28 .	1162·86	24·035	300·608	57·0070	26·38	·0522933	·000186357
July 29 .	1162·86	23·897	300·020	56·9050	25·12	·0521818	·000186350
„ 29 .	1162·86	24·022	299·993	56·8700	25·22	·0521723	·000186334
„ 29 .	1162·86	24·101	300·089	56·8690	25·22	·0521875	·000186325
*July 29 .	1162·86	25·010	299·841	56·6364	25·74	·0521411	·000186324
„ 29 .	1162·86	25·123	300·003	56·6404	25·75	·0521678	·000186311
Means =	—	—	300·092	—	—	·0521906	·000186334

* Apparatus allowed to cool between the two sets of observations.

Confirmatory Series.

As the determinations of the coefficient of expansion below 187° C. had all been made with the old set of steel tubes, whereas the determinations at temperatures above 187° C. had all been made after the apparatus had been taken down and re-erected with the new set of steel tubes, a short series of observations were taken to confirm the earlier results. Some observations were also taken at intermediate temperatures, but no use was made of these in evaluating an equation, as time did not permit of obtaining the steady state of temperature secured in runs of longer duration.

The following is a summary of the results, reduced to 0° C. :—

Date.	t_2.	${}_0\alpha_2 \times t_2$.	${}_0\alpha_2$ observed.	${}_0\alpha_2$ calculated.
July 19, 1909	136·270	·024898	·00018271	·00018272
„ 19, 1909	150·255	·027499	·00018302	·00018299
„ 19, 1909	168·009	·030804	·00018335	·00018336
July 20, 1909	81·140	·014748	·00018176	·00018173
„ 20, 1909	240·933	·044577	·00018502	·00018502

The results were not worked out beyond the limits of accuracy of the readings, but the agreement obtained with the equation calculated from the previous observations (see below, p. 22) was considered sufficient to show that no systematic change had occurred in the working of the apparatus.

In order to be able to reduce the observations with certainty to 0° C., and to obtain a direct value for the fundamental interval without extrapolation, it was necessary to take a series of observations with the cold column at a temperature as near 0° C. as possible. This point has already been explained in a previous section. By surrounding one side of the iron rectangle containing the cold column with a jacket of melting ice, it was found possible to reduce the temperature to between 2° C. and 2°·5 C. By further cooling the cold column to −10° C. with a freezing mixture of ice and salt, while the hot column remained at the atmospheric temperature of 16° C., it was possible to obtain a good approximation to the coefficient at 0° C. These observations entailed much greater difficulty in manipulation than the two previous series, but were valuable as giving direct evidence with regard to the expansion between −10° C. and +20° C.

Series III.—Observations from −10° C. *to* 100° C.

Date.	H_2.	dH.	t_1.	t_2.	h.	t.	${}_0\alpha_2 \times t_2$.	${}_0\alpha_2$.
(1) Observations, 2°·5 C. to 38° C.								
1910								
Jan. 5 . . .	1159·17	2·82	2·569	37·890	7·3716	14·42	·0068576	·000180986
,, 5 . . .	1159·17	2·82	2·528	37·933	7·3939	14·41	·0068696	·000181096
,, 5 . . .	1159·17	2·82	2·612	37·886	7·3680	14·81	·0068620	·000181120
Jan. 6 . . .	1159·17	2·82	2·346	37·259	7·2910	13·50	·0067481	·000181113
Jan. 7 . . .	1159·17	2·82	2·410	37·719	7·3733	12·10	·0068325	·000181144
,, 7 . . .	1159·17	2·82	2·464	37·787	7·3763	12·67	·0068445	·000181127
Means =	—	—	—	37·746	—	—	·0068357	·000181097
(2) Observations, 2°·5 C. to 68°·5 C.								
1910								
Jan. 5 . . .	1159·47	3·18	2·388	68·798	13·8308	14·87	·0124879	·000181515
,, 5 . . .	1159·47	3·18	2·534	68·968	13·8385	14·50	·0125222	·000181565
Jan. 6 . . .	1159·47	3·18	2·199	68·959	13·9040	14·04	·0125188	·000181540
,, 6 . . .	1159·47	3·18	2·328	69·036	13·8971	14·00	·0125364	·000181592
Jan. 7 . . .	1159·47	3·18	2·481	68·106	13·6671	12·83	·0123640	·000181540
,, 7 . . .	1159·47	3·18	2·587	68·223	13·6628	12·41	·0123808	·000181475
Means =	—	—	—	68·682	—	—	·0124683	·000181538

Series III.—Observations from $-10°$ *C. to* $100°$ *C.* (continued).

Date.	H_2.	dH.	t_1.	t_2.	h.	t.	${}_0\alpha_2 \times t_2$.	${}_0\alpha_2$.
			(3) Observations, 2°·5 C. to 100° C.					
1910								
Jan. 5 . . .	1160·04	3·75	2·494	100·778	20·4250	14·87	·0183475	·000182059
„ 5 . . .	1160·04	3·75	2·576	100·703	20·3932	14·91	·0183342	·000182062
Jan. 6 . . .	1160·04	3·75	2·338	100·547	20·4081	14·65	·0183040	·000182042
„ 6 . . .	1160·04	3·75	2·407	100·626	20·4090	14·56	·0183180	·000182040
Jan. 7 . . .	1160·04	3·75	2·609	100·442	20·3252	14·16	·0182819	·000182015
„ 7 . . .	1160·04	3·75	2·703	100·505	20·3241	14·11	·0182987	·000182068
Means =	—	—	—	100·600	—	—	·0183140	·000182048
			(4) Observations, −10° C. to + 16° C.				${}_1\alpha_2\,(t_2 - t_1)$.	${}_1\alpha_2$.
1910								
Jan. 10 . .	1158·66	2·37	−10·403	16·012	5·5220	14·33	·0047765	·00018082
„ 11 . .	1158·66	2·37	−10·498	16·205	5·5870	13·70	·0048332	·00018098
Means =	—	—	−10·450	+ 16·109	—	—	·0048049	·00018090

The last observations give the mean coefficient from $-10°\cdot5$ C. to $+16°\cdot1$ C., which is practically the same as the actual coefficient at $2°\cdot8$ C. The values are expressed by formula (2) in terms of the volume at $-10°\cdot450$ C. When expressed in terms of the volume at 0° C. by formula (3), the values become

$$\text{Means} \quad . \quad . \quad . \quad [{}_1\alpha_2]_0\,(t_2-t_1) = 0\cdot0047958. \qquad [{}_1\alpha_2]_0 = 0\cdot000180572.$$

8. *Empirical Formulæ for the Expansion of Mercury.*

An empirical formula representing the first two series was calculated in the first instance by the method of least squares. This method was adopted by WÜLLNER and BROCH in reducing REGNAULT'S observations, and was fairly appropriate in that case, because the main source of error lay in measuring the small difference of level in the gauge. In the present series of experiments, the fact that the difference of level in the gauge could not be read nearer than 0·001 cm. was an important limitation of accuracy at low temperatures, when the difference of level was small. But at temperatures between 200° C. and 300° C., where the difference of level was 40 to 60 cm., the possible errors in the measurement of the length and temperature of the hot columns became more important, and the order of accuracy was limited in a different way, namely, as a fraction of the whole quantity measured. For low temperatures, the differences between the observed and calculated values of the expansion itself were the best criterion of accuracy; but for high temperatures, the corresponding differences between the observed and calculated values of the *coefficient* of expansion appeared to be a better guide in the selection of an equation. The formula obtained by the method of least squares was accordingly modified from this point of view, but the modifications required were so slight as to be almost within the limits of experimental error.

The following formula was finally adopted to represent the value of the mean coefficient ${}_0\alpha_t$ between 0° C. and t° C. in terms of the volume at 0° C. :—

$$ {}_0\alpha_t = [1805553+12444\,(t/100)+2539\,(t/100)^2]\times 10^{-10}. \quad . \quad . \quad . \quad . \quad (8) $$

The value of the fundamental coefficient ${}_0\alpha_{100}$ given by this formula is

$$ {}_0\alpha_{100} = 0{\cdot}0001820536. $$

It is, unfortunately, impossible to represent the results satisfactorily over the whole range by a linear formula for the mean coefficient of expansion, because the rate of increase of the mean coefficient is more than twice as great at 300° C. as at 0° C. But for approximate work the following simple formula for the mean coefficient may be sufficiently exact to be of use :—

$$ {}_0\alpha_t = (18006+2t)\times 10^{-8}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9) $$

This formula gives results which are practically correct at 100° C. and 200° C., and which do not differ from formula (8) by so much as 0°·05 C. at 50° C. and at 150° C. But the value of the mean coefficient is about 1 in 400 too low in the neighbourhood of 0° C. and 300° C.

For convenience of comparison with the above formula (8), the observations of Series I. and II. (which were reduced to 20° C. in the first instance, and expressed in terms of the volume at 20° C.) are here reduced to 0° C., and expressed in terms of the volume at 0° C., by multiplying the values of V_t/V_{20} [namely, $1+{}_{20}\alpha_t\times(t-20)$] given in the tables, by the value of V_{20}/V_0 (namely, 1·00361632), given by the formula of comparison. This reduction will not introduce any error in the comparison of the observed results with those calculated by the formula.

TABLE I.—Comparison of Results with Formula (8).

Series and Number.	Temperature, t_2.	Expansion $(V_t - V_0)/V_0$.		Difference, $C - O \times 10^7$.	Mean coefficient 0° to t_2.		Difference, $C - O \times 10^9$.
		Observed.	Calculated.		Observed.	Calculated.	
III. (4)	{ −10·450 / +16·109 }	·0047958	·0047974	+16	·000180572	·000180632	+60
III. (1)	37·746	·0068357	·0068343	−14	·000181097	·000181061	−36
I. (1)	68·649	·0124629	·0124618	−11	·000181545	·000181529	−16
III. (2)	68·682	·0124683	·0124678	− 5	·000181537	·000181530	− 7
I. (2)	100·314	·0182626	·0182631	+ 5	·000182054	·000182059	+ 5
III. (3)	100·600	·0183140	·0183156	+16	·000182048	·000182064	+16
I. (3)	116·858	·0213079	·0213098	+19	·000182340	·000182356	+16
I. (4)	150·375	·0275216	·0275187	−29	·000183020	·000183001	−19
I. (5) } II. (1) }	187·460	·0344547	·0344514	−33	·000183798	·000183781	−17
II. (2)	221·145	·0408120	·0408121	+ 1	·000184549	·000184549	+ 0
II. (3)	259·904	·0482078	·0482134	+56	·000185483	·000185505	+22
II. (4)	300·092	·0559956	·0559900	−56	·000186595	·000186576	−19

The observations are arranged in order of temperature. The first column gives the series and number corresponding to the previous tables of observations. The second gives the temperature t_2 of the hot column, except in the case of the first line, Observation III. (4), where the lower limit was −10°·450 C. in place of 0° C., and both limits are given. The third and fourth columns give the observed and calculated values of the expansion $(V_t - V_0)/V_0$ between 0° C. and t_2, except in the first line, where the expansion $(V_2 - V_1)/V_0$ between t_1 and t_2 is given instead. The fifth column gives the differences between the calculated and observed values of the expansion multiplied by 10^7. The last three columns give the observed and calculated values of the mean coefficient from 0° C. to t_2, and the difference $(C - O) \times 10^9$, except in the first line, where the mean coefficients are from t_1 to t_2.

9. *Order of Accuracy of the Results.*

In comparing the results with the formula, it must be observed that the differences in Table I. are all calculated to one figure beyond the limit of accuracy of observation, namely, 0·001 cm., which corresponds to 10 in the difference between the calculated and observed values of the expansion. Taking the observations at and below 100° C., the mean deviation of the observed expansion from the formula is only 11, which corresponds with the limit of accuracy of reading. The corresponding differences in the values of the mean coefficients, given in the last column, are here without significance, because it was obviously impossible to measure a short column of only 5·5 cm. to an order of accuracy of 1 in 20,000 under the conditions of the experiment.

Taking the observations above 100° C., the mean deviation of the values of the mean *coefficient* of expansion from the formula is 15, which corresponds to an order of accuracy of 1 in 12,000. The differences between the observed and calculated values of the expansion itself are of the same relative order, and increase in absolute magnitude, as one would naturally expect, with increase of temperature.

None of the observations at 100° C. or below differ from the formula by so much as 0·002 cm. or 0°·01 C. Only one of the observations above 100° C. differs from the formula by as much as 1 in 8,500. We may fairly conclude that the formula represents the results with an order of accuracy of 0°·01 C. at temperatures below 100° C., and with an order of accuracy of 1 in 10,000 above 100° C. Since positive and negative differences occur almost alternately, and are little, if at all, greater than might naturally be expected from the limits of accuracy of the various readings, it does not appear that any great advantage could be gained by the adoption of a more complicated formula, or by any more elaborate reduction or repetition of the experiments.

The mean deviation of the individual observations at each point is about twice as great as the deviation of the mean results from the formula. The individual observations are affected by accidental errors of refraction through the glass of the gauge tubes, and by errors of lag, which would disappear to some extent in the means. Correction for lag would have made the observations agree with each other much better in most cases, but the correction could not always be applied with certainty, and it was therefore preferably omitted from the tables.

10. *Comparison with Previous Results.*

It may be of interest to compare the results of the present investigation, as expressed by formula (8), with some of the formulæ which have previously been employed to represent the expansion of mercury.

REGNAULT assumed a linear formula for the mean coefficient, namely,

$$_0\alpha_t = \{179007 + 2523\,(t/100)\} \times 10^{-9}.$$

He appears to have relied chiefly on the observations at the higher temperatures, and the formula does not represent his observations satisfactorily at temperatures below 150° C.

BROCH, in reducing REGNAULT'S results, assumed a parabolic formula of the same type as formula (8) for the mean coefficient. He also introduced a correction for the conduction of heat along the cross tubes, which were not quite horizontal in REGNAULT'S fourth series of observations, in order to reconcile the results of the fourth series with those of the first three. The formula deduced by BROCH was as follows:—

$$_0\alpha_t = \{1817920 + 175\,(t/100) + 3511{\cdot}6\,(t/100)^2\} \times 10^{-10}.$$

CHAPPUIS gave a formula of a similar type, to represent the results of his observations by the weight thermometer method between 0° C. and 100° C.

$$_{0}\alpha_{t} = \{1816904 \cdot 1 - 2951 \cdot 266\,(t/100) + 11456 \cdot 2\,(t/100)^2\} \times 10^{-10}.$$

This formula has been extrapolated by EUMORFOPOULOS ('Roy. Soc. Proc.,' A, vol. 81, p. 339, 1908), but extrapolation in such a case would be somewhat unreliable.

The following table gives a short comparison of the above formulæ with formula (8), showing the values of the mean coefficient multiplied by 10^9, together with the differences from formula (8) :—

TABLE II.—Comparison of Formulæ.

Temperature.	CALLENDAR and Moss (8).	BROCH.	Difference.	CHAPPUIS.	Difference.	REGNAULT.	Difference.
° C.							
40	181094	181855	+ 761	181755	+ 661	180025	− 1069
100	182054	182161	+ 107	182541	+ 487	181530	− 524
140	182795	182506	− 289	(183323	+ 528)	182536	− 259
200	184060	183232	− 828	(185683	+ 1623)	184055	− 5
240	185004	183857	− 1147	(187591	+ 2587)	185063	+ 59
300	186574	185005	− 1569	(191116	+ 4542)	186577	+ 3

A comparison of these differences with those given in Table I. on p. 24 in terms of the same unit, illustrates the state of uncertainty which existed with regard to the expansion of mercury in the year 1907, and may be taken as sufficient excuse for the publication of the present work.

A similar comparison is shown graphically in a slightly different manner by the curves in fig. 8. Since it would be impossible to plot the expansion itself graphically on an adequate scale, even by the copper-plate method employed by REGNAULT, the quantity plotted in fig. 8 is the difference of the expansion from lineality, or the difference ($\alpha - 0 \cdot 000182054$) of the mean coefficient α from the fundamental coefficient multiplied by t. The heavy line with the large circles and crosses ⊕ represents the results of the present series of observations. The deviations from the curve on this scale scarcely exceed the thickness of the line. The dots surrounded by small circles ⊙ represent REGNAULT'S actual observations. The broken lines represent the formulæ of REGNAULT, WÜLLNER, and BROCH. It is evident that the curve representing our results also represents REGNAULT'S observations, *as reduced by himself*, much better than they are represented by any of the other three formulæ. The difference between the curves given by REGNAULT and WÜLLNER arises chiefly from the uncertainty already alluded to on p. 9 in reducing REGNAULT'S observations to 0° C. The great deviation of BROCH'S curve from the others at high temperatures appears to arise chiefly from the correction which he introduced in the endeavour to reconcile

REGNAULT'S fourth series with the first three. This correction produces a much larger deviation than the original discrepancy. It must be admitted, however, that a deviation of the type assumed by BROCH was quite possible so far as the evidence of

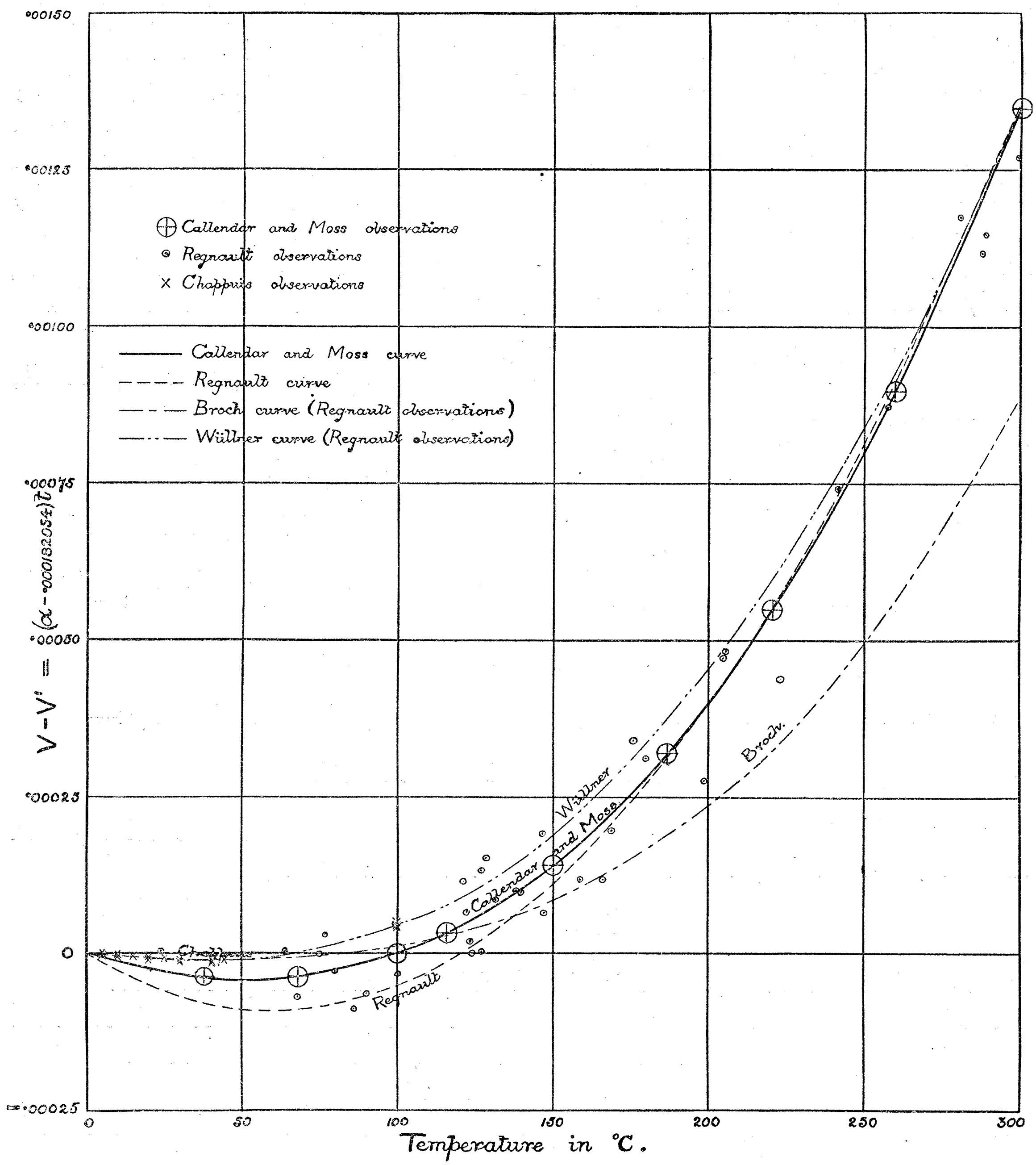

Fig. 8.

the observations went. It is satisfactory to find that the error was not so serious as he supposed, and that the results of the present investigation are in such good general agreement with REGNAULT'S work.

11. *Corrected Value of the Boiling-Point of Sulphur.*

The preliminary results published in the "Note on the Boiling-Point of Sulphur" in September, 1909, were affected by a small error in the fundamental interval, which at that time was uncertain, as the observations of Series III. had not then been taken. As pointed out in the note in question, a small error of this type is practically without effect on the result, owing to the manner in which the fundamental coefficient enters into the formula for the correction of the gas thermometer. The final value of the boiling-point of sulphur given in the note is not changed by more than 0°·01 C. by the error in the fundamental interval itself. Unfortunately the corresponding error in the coefficient b, though much smaller, produces a larger error in the result, namely, 0°·06 C., but this is still within the limits of error of the gas thermometer.

The following are the corrected values:—

The final corrected values of the ratios of the densities of mercury at 0° C., 100° C., and 184° C., given by formula (8), are as follows:—

$$D_0/D_{100} = 1\cdot0182054, \qquad D_0/D_{184}\ 1\cdot0338016.$$

The observations taken with the weight thermometer in March, 1900, as reduced by EUMORFOPOULOS, assuming BROCH's reduction of REGNAULT's observations, gave the following values of the coefficients expressing the expansion of the bulb:—

$$a = 2387 \times 10^{-8}, \qquad b = 0\cdot42 \times 10^{-8}.$$

Our final corrected values of the expansion of mercury give the following:—

$$a = 2377 \times 10^{-8}, \qquad b = 1\cdot37 \times 10^{-8}.$$

The correction to be added to the results of EUMORFOPOULOS for the boiling-point of sulphur, calculated by the formula given in the previous note, is

$$dt = +1^\circ\cdot03 \text{ C.}$$

in place of $dt = +0^\circ\cdot97$ C., as previously found by the preliminary reduction.

Strictly speaking, this correction applies only to the gas-thermometer observations taken with the same bulb as that used for the weight-thermometer determinations. The value of the boiling-point of sulphur found with this particular bulb in March, 1900, was $t = 443^\circ\cdot48$ C. The addition of the above correction would raise this result to $t = 444^\circ\cdot51$ C. The final mean obtained by EUMORFOPOULOS from observations with other bulbs, of which the expansion was not directly determined, was $t = 443^\circ\cdot58$ C. This would raise the corrected value of the boiling-point to $t = 444^\circ\cdot61$ C. But since the later bulbs were not treated in exactly the same manner as the first bulb, it is probable that greater weight should be attached to the first result. The uncertainty of the gas-thermometer determinations at this point is of the order of 0°·1 C., and there does not seem to be any sufficient reason for changing the final value of the boiling-point of sulphur on the scale of the constant-pressure air or nitrogen thermometer from that given in the previous note and assumed for so many years, namely, $t = 444°\cdot53$ C.

12. *Explanation of the Tables of Expansion.*

The accompanying tables of the expansion of mercury from $-30°$ C. to $309°$ C., together with the table of differences on the opposite page, make it easy to calculate the expansion from $0°$ C. to any other temperature within the given limits. If one of the limits be not $0°$ C., the volume at each limit must be found, and the difference taken.

The following examples will make the use of the tables clear:—

(1) To find the expansion from 0° C. to 221°·145 C.—

Expansion from 0° C. to 221° C.		0·0407846
Difference for 0°·1 C. at 220° C.		190
,, ,, 0°·04 C. ,, 220° C.		76
,, ,, 0°·005 C. ,, 220° C.		10
Expansion from 0° C. to 221°·145 C.		0·0408122

The values found in this way from the tables will, in general, be correct to 1 in the last figure, or 0°·001 C., as given by formula (8).

(2) To find the expansion from $-10°\cdot450$ C. to $+16°\cdot109$ C.—

Volume at −10° C.		0·9981957
Difference for −0°·4 C. at −10° C.	−	722
,, ,, −0°·050 C. ,, −10° C.	−	90
Volume at −10°·450 C.		0·9981145
Volume at +16° C.		1·0028922
Difference for 0°·1 C. at 16° C.	+	181
,, ,, 0°·009 C. ,, 16° C.	+	16
Volume at +16°·109 C.		1·0029119
Volume at −10°·450 C.		0·9981145
Expansion between limits		0·0047974

TABLE III.—Expansion of Mercury from −30° C. to 309° C.

$$\frac{V}{V_0} = 1 + {}_0\alpha_t t = 1 + 1805553\left(\frac{t}{100}\right)10^{-8} + 12444\left(\frac{t}{100}\right)^2 10^{-8} + 2539\left(\frac{t}{100}\right)^3 10^{-8}.$$

Temperature. t° C.	Degrees. 0	1	2	3	4	5	6	7	8	9	Temperature. t° C.
−30	·99 45939	47738	49537	51336	53135	54935	56735	58535	60335	62136	−30
−20	63937	65738	67539	69340	71142	72944	74746	76548	78351	80154	−20
−10	81957	83760	85563	87367	89171	90975	92780	94584	96389	98195	−10
0	1·00 00000	01806	03612	05418	07224	09031	10838	12645	14452	16260	0
10	18068	19876	21685	23494	25303	27112	28922	30732	32542	34352	10
20	36163	37974	39785	41597	43409	45221	47033	48846	50659	52472	20
30	54285	56099	57913	59728	61543	63358	65173	66989	68805	70621	30
40	72437	74254	76072	77889	79707	81525	83344	85162	86981	88801	40
50	90621	92441	94261	96082	97903	99724	$\overline{01546}$	$\overline{03368}$	$\overline{05190}$	$\overline{07013}$	50
60	1·01 08836	10659	12483	14307	16132	17957	19782	21607	23433	25259	60
70	27086	28913	30740	32567	34395	36224	38052	39881	41711	43541	70
80	45371	47201	49032	50863	52695	54527	56359	58192	60025	61859	80
90	63693	65527	67362	69197	71032	72868	74705	76541	78378	80216	90
100	82054	83892	85730	87570	89409	91249	93089	94930	96771	98613	100
110	1·02 00455	02297	04140	05983	07826	09670	11515	13360	15205	17051	110
120	18897	20744	22591	24438	26286	28134	29983	31832	33682	35532	120
130	37383	39234	41085	42937	44789	46642	48495	50349	52203	54058	130
140	55913	57769	59625	61481	63338	65196	67054	68912	70771	72630	140
150	74490	76350	78211	80072	81934	83796	85659	87522	89835	91249	150
160	93114	94979	96845	98711	$\overline{00578}$	$\overline{02445}$	$\overline{04312}$	$\overline{06180}$	$\overline{08049}$	$\overline{09918}$	160
170	1·03 11788	13658	15529	17400	19272	21144	23016	24889	26763	28637	170
180	30512	32387	34263	36140	38016	39894	41772	43650	45529	47409	180
190	49289	51169	53050	54932	56815	58697	60580	62464	64349	66234	190
200	68119	70005	71892	73779	75667	77555	79444	81334	83224	85114	200
210	87005	88897	90789	92682	94576	96470	98364	$\overline{00259}$	$\overline{02155}$	$\overline{04051}$	210
220	1·04 05948	07846	09744	11642	13541	15441	17342	19243	21144	23046	220
230	24949	26853	28757	30661	32566	34472	36379	38286	40193	42101	230
240	44010	45920	47830	49741	51652	53564	55476	57390	59304	61218	240
250	63133	65049	66965	68882	70799	72718	74637	76556	78476	80397	250
260	82318	84240	86163	88087	90011	91935	93861	95787	97713	99641	260
270	1·05 01569	03497	05426	07356	09287	11218	13150	15083	17016	18950	270
280	20885	22820	24756	26692	28630	30568	32506	34446	36386	38327	280
290	40268	42210	44153	46097	48041	49986	51931	53878	55825	57772	290
300	59721	61670	63620	65570	67522	69474	71426	73380	75334	77289	300

TABLE of Differences for Fractions of 1° C.

Temperature. $t°$ C.	Differences for tenths of 1° C. × 10^7. Tenths of a ° C. ·1	·2	·3	·4	·5	·6	·7	·8	·9	Temperature. $t°$ C.	Mean coefficient of expansion between 0° C. and $t°$ C.
−30	180	360	540	720	900	1080	1260	1440	1620	−30	·000180205
−20	180	360	541	721	901	1081	1261	1442	1622	−20	0317
−10	180	361	541	722	902	1083	1263	1443	1624	−10	0433
0	181	361	542	723	903	1084	1265	1445	1626	0	0555
10	181	362	543	724	905	1086	1267	1448	1629	10	0682
20	181	362	544	725	906	1087	1269	1450	1631	20	0814
30	182	363	545	726	908	1089	1271	1452	1634	30	0951
40	182	364	546	727	909	1091	1273	1455	1636	40	1094
50	182	364	546	729	911	1093	1275	1457	1639	50	1241
60	182	365	547	730	912	1095	1277	1460	1642	60	1393
70	183	366	549	731	914	1097	1280	1463	1646	70	1551
80	183	366	550	733	916	1099	1283	1466	1649	80	1713
90	184	367	551	734	918	1102	1285	1469	1652	90	1881
100	184	368	552	736	920	1104	1288	1472	1656	100	2054
110	184	369	553	738	922	1107	1291	1475	1660	110	2231
120	185	370	555	739	924	1109	1294	1479	1664	120	2414
130	185	371	556	741	926	1112	1297	1482	1668	130	2602
140	186	372	557	743	929	1115	1300	1486	1672	140	2795
150	186	372	559	745	931	1117	1304	1490	1676	150	2993
160	187	373	560	747	934	1120	1307	1494	1681	160	3196
170	187	374	562	749	936	1123	1311	1498	1685	170	3405
180	188	376	563	751	939	1127	1314	1502	1690	180	3618
190	188	377	565	753	941	1130	1318	1506	1695	190	3836
200	189	378	567	755	944	1133	1322	1511	1700	200	4060
210	189	379	568	758	947	1137	1326	1515	1705	210	4288
220	190	380	570	760	950	1140	1330	1520	1710	220	4522
230	191	381	572	762	953	1144	1334	1525	1715	230	4761
240	191	382	574	765	956	1147	1339	1530	1721	240	5004
250	192	384	576	767	959	1151	1343	1535	1727	250	5253
260	192	385	577	770	962	1155	1347	1540	1732	260	5507
270	193	386	579	773	966	1159	1352	1545	1738	270	5766
280	194	388	581	775	969	1163	1357	1551	1744	280	6030
290	195	389	584	778	973	1167	1362	1556	1751	290	6299
300	195	390	586	781	976	1171	1367	1562	1757	300	6574

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The actual coefficient at any temperature is seldom required with a high order of accuracy. It may be obtained from the tables with sufficient accuracy by taking the difference of the volumes for a range of 5° C. on either side of the point where the coefficient is required, and dividing by 10. *E.g.*, to find the coefficient at 300° C., we have

Volume at 305° C. = 1·0569474,
,, ,, 295° C. = 1·0549986,
Difference/10 = Coefficient of expansion at 300° C. = 0·00019488.

[*Note added February* 13, 1911.—It should be observed that the expansion of mercury is here expressed in terms of the scale of temperature, based on the platinum resistance thermometer, proposed by CALLENDAR ('Phil. Mag.,' December, 1899, p. 519) at the meeting of the British Association at Dover. This scale assumes the formula given on p. 7 above for reducing the readings of a platinum thermometer to the gas-scale, and is equivalent to assuming the value 444°·53 C. for the boiling-point of sulphur. It was admitted that this value might require a correction between +0°·3 C. and 0°·5 C. to reduce it to the absolute scale, but, as this correction depended on the extrapolation of experiments between 0° C. and 100° C., it was considered inadvisable to alter the existing standard scale of platinum thermometry until further experiments had been made with helium and argon at high temperatures. Many writers now adopt values ranging from 444·8 to 445·0 for the boiling-point of sulphur. This may lead to some confusion unless a definite convention is established. Until the correction to the absolute scale has been determined with greater precision it would be preferable to retain the old scale.]

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